International Mathematical Olympiad Amsterdam 2011

Language: **English**

Day: **1**

Monday, July 18, 2011

Problem 1. Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

Problem 2. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.
Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x + y) \leq y f(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.