Problems and Solutions of Geometry Unbound

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1. Problem 1.2.2 (USAMO 1994/3)

A convex hexagon $ABCDEF$ is inscribed in a circle such that $AB = CD = EF$ and diagonals AD, BE, CF are concurrent. Let P be the intersection of AD and CE.
Prove that

$$\frac{CP}{PE} = \left(\frac{AC}{CE}\right)^2$$

Solution: Let $\theta = \angle ACB$, $\alpha = \angle BDC$, $\beta = \angle DFE$, $\gamma = \angle FBA$. Then $\angle EPA = \angle EDB = \angle CPD = 2\theta + \gamma$ and $\angle PAE = \angle DBE = \angle DCP = \beta$, so $\triangle EPA \sim \triangle EDB \sim \triangle DPC$. Therefore

$$\frac{CP/CD}{PE/AE} = \frac{AP/AE}{PE/AE} = \frac{AP}{PE} = \frac{BD}{DE}.$$

Also $\angle ECA = \angle DOC = \angle EDO = \theta + \gamma$ and $\angle AEC = \angle CDO = \angle OED = \theta + \alpha$, so $\triangle ACE \sim \triangle COD \sim \triangle ODE$. (In fact, all six triangles given by O and two adjacent vertices of hexagon $ABCDEF$ are similar to ACE, by analogous angle-chasing.) Finally, $\triangle ACE \cong \triangle BDF$ as $ABCD$, $CDEF$, $EFAB$ are all isosceles trapezoids. Therefore

$$\frac{CP}{PE} = \frac{CD}{AE}\frac{BD}{DE} = \frac{OD}{CE}\frac{AC}{DE} = \frac{AC}{CE}\frac{OD}{DE} = \left(\frac{AC}{CE}\right)^2.$$

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2. Problem 1.2.3 (IMO 1990/1)

Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line of E to the circle through D, E, M intersects the lines BC and AC at F and G, respectively. If $AM/AB = t$, find EG/EF in terms of t.

Solution: Let N be the second intersection of the circle through A, B, C, D with the circle through D, E, M. Note $\angle NEG = \angle NDE = \angle NDC = \angle NBC = 180 - \angle NAC = \angle NAG$; therefore N, G, A, and E are concyclic, so $\angle NGE = \angle NAE = \angle NAM$. We also have $\angle NMA = \angle NME = \angle NEG$, so $\triangle NAM \sim \triangle NGE$; therefore

$$\frac{EG}{AM} = \frac{NG}{NA}.$$

As $\angle NBF = \angle NBC = \angle NEG = \pi - \angle NEF$, N, B, E, F are concyclic, so $\angle NFG = \angle NFE = \angle NBE = \angle NBA$; as $\angle NGF = \angle NGE = \angle NAE = \angle NAB$, $\triangle NGF \sim \triangle NAB$, so

$$\frac{GF}{AB} = \frac{NG}{NA}.$$

These two equations give us $EG/GF = AM/AB = 1/t$; simple algebra gives

$$\frac{EG}{EF} = \frac{t}{1-t}$$

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3. Problem 1.3.2

Two circles intersect at points A and B. An arbitrary line through B intersects the first circle again at C and the second circle again at D. The tangents to the first circle at C and the second at D intersect at M. Through the intersection of AM and CD, there passes a line parallel to CM and intersecting AC at K. Prove that BK is tangent to the second circle.

Solution: Note that $\angle DMC = \angle MDC + \angle DCM = \angle MDB + \angle BCM = \angle DAB + \angle BAC = \angle DAC$, so points A, C, D, and M are concyclic. Let $P = AM \cap CD$; then $\angle KAB = \angle CAB = \angle MCB = \angle MCP = \angle KPC = \angle KPB$, so points A, K, B, P are concyclic. Now

$$\angle KBD = \angle KBP = \angle KAP = \angle CAM = \angle CDM = \angle BDM = \angle BAD;$$

therefore BK is tangent to the second circle.

4. Problem 1.3.3

Let C_1, C_2, C_3, C_4 be four circles in the plane. Suppose that C_1 and C_2 intersect at P_1 and Q_1, C_2 and C_3 intersect at P_2 and Q_2, C_3 and C_4 intersect at P_3 and Q_3, and C_4 and C_1 intersect at P_4 and Q_4.

Show that if P_1, P_2, P_3, and P_4 lie on a line or circle,then Q_1, Q_2, Q_3, and Q_4 also lie on a line or circle.

Solution: Suppose P_1, P_2, P_3, P_4 lie on a line or circle; then $\angle P_4P_1P_2 = \angle P_4P_3P_2$, so $\angle P_4P_1P_2 + \angle P_2P_3P_4 = 0$. We have

$$\angle Q_1Q_2Q_3 = \angle Q_1Q_2P_2 + \angle P_2Q_2Q_3 = \angle Q_1P_1P_2 + \angle P_2P_3Q_3$$
$$\angle Q_3Q_4Q_1 = \angle P_4Q_4Q_1 + \angle Q_3Q_4P_4 = \angle P_4P_1Q_4 + \angle Q_3P_3P_4$$

so $\angle Q_1Q_2Q_3 + \angle Q_3Q_4Q_1 = \angle P_4P_1P_2 + \angle P_2P_3P_4 = 0$.Therefore Q_1, Q_2, Q_3, Q_4 lie on a line or circle.

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5. Problem 1.4.1 (IMO 1994/2) Let ABC be an isosceles triangle with $AB = AC$. Suppose that

1. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;
2. Q is an arbitrary point on the segment BC different from B and C;
3. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

Solution: First, suppose $OQ \perp EF$. Then $\angle EBO = \angle EQO = \angle FQO = \angle FCO = \pi/2$, so quadrilaterals $BQOE$ and $FQOC$ are cyclic. Therefore $\angle FEO = \angle QEO = \angle QBO = \angle CBO = \angle BCO = \angle QCO = \angle QFO = \angle EFO$, so $OE = OF$; since $OQ \perp EF$, $QE = QF$.

Now suppose $QE = QF$, but OQ is not perpendicular to EF. Construct $E'F'$ through Q perpendicular to OQ with E' on the ray AB and F' on the ray AC; then by the first part $QE' = QF'$. Since $QE = QF$ and $\angle EQE' = \angle FQF'$, $\triangle QEE' \cong \triangle QFF'$. But then $\angle EE'F' = \angle EE'Q = \angle FF'Q = \angle FF'E'$, so $EE' \parallel FF'$, impossible as then $AB \parallel AC$. So $OQ \perp EF$.

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6. Problem 2.1.1

Suppose the cevians AP, BQ, CR meet at T.

Prove that
$$\frac{TP}{AP} + \frac{TQ}{BQ} + \frac{TR}{CR} = 1$$

Solution:

Let $K = [ABC]$. Then $TP/AP = [TBC]/K$, $TQ/BQ = [TCA]/K$, $TR/CR = [TAB]/K$, so
$$\frac{TP}{AP} + \frac{TQ}{BQ} + \frac{TR}{CR} = \frac{[TBC] + [TCA] + [TAB]}{K} = \frac{[ABC]}{K} = 1.$$

7. Problem 2.1.3 (Hungary-Israel, 1997)

The three squares ACC_1A'', $ABB_1'A'$, $BCDE$ are constructed externally on the sides of a triangle ABC. Let P be the center of $BCDE$. Prove that the lines $A'C$,$A''B$, PA are concurrent.

Solution: Let A_1 be the foot of the perpendicular from A'' to AB, and C_1 the foot of the perpendicular from A'' to BC; then

$$\frac{\sin\angle ABA''}{\sin\angle A''BC} = \frac{A''A_1/BA''}{A''C_1/BA''} = \frac{A''A_1}{A''C_1} = \frac{b\cos A}{b\sqrt{2}\cos(C+\pi/4)} = \frac{\cos A}{\cos C - \sin C}.$$

(We take $A''A_1 > 0$ when A'' and C are on the same side of A_1, otherwise $A''A_1 < 0$; similarly for $A''C_1$.) Similarly

$$\frac{\sin\angle BCA'}{\sin\angle A'CA} = \frac{c\sqrt{2}\cos(B+45)}{c\cos A} = \frac{\cos B - \sin B}{\cos A}.$$

Finally, let C_2 be the foot of the perpendicular from P to AC and B_2 the foot of the perpendicular from P to AB; then

$$\frac{\sin\angle CAP}{\sin\angle PAB} = \frac{PC_2/AP}{PB_2/AP} = \frac{PC_2}{PB_2} = \frac{(a/\sqrt{2})\cos(C+45)}{(a/\sqrt{2})\cos(B+45)} = \frac{\cos C - \sin C}{\cos B - \sin B}.$$

Therefore

$$\frac{\sin\angle ABA''}{\sin\angle A''BC}\frac{\sin\angle BCA'}{\sin\angle A'CA}\frac{\sin\angle CAP}{\sin\angle PAB} = \frac{\cos A(\cos B - \sin B)(\cos C - \sin C)}{(\cos C - \sin C)\cos A(\cos B - \sin B)} = 1,$$

so AP, BA'', CA' concur by Trig Ceva.

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8. Problem 2.1.4 (Răzvan Gelca) r

*Let*ABC*beatriangleand*D,E, F*thepointswheretheincircletouchesthesides*BC,CA,AB, *respectively.Let*
FD,DE*respectively.Showthatthelines*AM,BN,CP*intersectifandonlyifthelines*DM,EN,FP*intersect.*

Solution:*From*M*dropperpendiculars*MR,MQ*to*AB,AC*respectively.Then*$\triangle FRM \sim \triangle EQM$, as $\angle RFM = \angle AFE = \angle FDE = \angle FEA = \angle MEQ$; therefore

$$\frac{\sin\angle BAM}{\sin\angle MAC} = \frac{RM/MA}{QM/MA} = \frac{RM}{QM} = \frac{FM}{EM}.$$

Therefore

$$\frac{\sin\angle BAM}{\sin\angle MAC}\frac{\sin\angle ACP}{\sin\angle PCB}\frac{\sin\angle CBN}{\sin\angle NBA} = \frac{FM}{ME}\frac{EP}{PD}\frac{DN}{NF},$$

so DM, EN, FP concur if and only if AM, BN, CP do.

9. Problem 2.1.5 (USAMO 1995/3)

Given a nonisosceles, nonright triangle ABC inscribed in a circle with center O, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that $\triangle OAA_1$ is similar to $\triangle OA_2A$. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent.

Solution: Let G be the centroid and H the orthocenter of $\triangle ABC$. Then $\angle OAA_2 = \angle OA_1A = \angle A_1AH$, and $\angle BAO = \pi/2 - C = \angle HAC$, so $\angle BAA_2 = \angle A_1AC$. Similarly $\angle AA_2C = \angle BAA_2$, etc., so

$$\frac{\sin \angle BAA_2}{\sin \angle A_2AC} \frac{\sin \angle ACC_2}{\sin \angle C_2CB} \frac{\sin \angle CBB_2}{\sin \angle B_2BA} = \frac{\sin \angle A_1AC}{\sin \angle BAA_1} \frac{\sin \angle B_1BA}{\sin \angle CBB_1} \frac{\sin \angle C_1CB}{\sin \angle ACC_1} = 1$$

by Trig Ceva, since AA_1, BB_1, CC_1 concur at G. Therefore AA_2, BB_2, CC_2 concur as well. (Their point of concurrence is called the *isogonal conjugate* of G; see section 5.5.)

10. Problem 2.1.6

Given triangle ABC and points X, Y, Z such that $\angle ABZ = \angle XBC, \angle BCX = \angle YCA, \angle CAY = \angle ZAB$, prove that AX, BY, CZ are concurrent.

Solution: Let $\alpha = \angle ABZ = \angle XBC$, $\beta = \angle BCX = \angle YCA$, $\gamma = \angle CAY = \angle ZAB$. Drop perpendiculars XP, XQ from X to AB, AC respectively. Then

$$\frac{\sin \angle BAX}{\sin \angle XAC} = \frac{PX/XA}{QX/XA} = \frac{PX}{QX} = \frac{BX \sin(B-\beta)}{CX \sin(C-\gamma)} = \frac{\sin\gamma \sin(B-\beta)}{\sin\beta \sin(C-\gamma)}$$

by the Law of Sines. So

$$\frac{\sin \angle BAX}{\sin \angle XAC} \frac{\sin \angle ACZ}{\sin \angle ZCB} \frac{\sin \angle CBY}{\sin \angle YBA} = \frac{\sin\gamma \sin(B-\beta)}{\sin\beta \sin(C-\gamma)} \frac{\sin\beta \sin(A-\alpha)}{\sin\alpha \sin(B-\beta)} \frac{\sin\alpha \sin(C-\gamma)}{\sin\gamma \sin(A-\alpha)} = 1,$$

and AX, BY, CZ concur by Trig Ceva.

11. Problem 2.2.2

Let A, B, C be three points on a line. Pick a point D in the plane, and a point E on BD. Then draw the line through $AE \cap CD$ and $CE \cap AD$.
Show that this line meets the line AC in a point P that depends only on A, B, C.

Solution: Let $F = CE \cap AD$, $G = AE \cap CD$. Then AG, DB, CF concur (at E), so by Ceva's Theorem

$$\frac{AB}{BC}\frac{CG}{GD}\frac{DF}{FA} = 1.$$

Applying Menelaos to the points P, G, F on the sides of triangle ACD gives

$$\frac{AP}{PC}\frac{CG}{GD}\frac{DF}{FA} = -1.$$

Therefore $AB/BC = -AP/PC$, so $AC/PC = 1 + AP/PC = 1 - AB/BC$, and $PC = AC/(1 - AB/BC)$; therefore P depends only on A, B, and C.

12. Problem 2.2.3

Let A, B, C be three collinear points and D, E, F three other collinear points.
Let $G = BE \cap CF, H = AD \cap CF, I = AD \cap CE$. If $AI = HD$ and $CH = GF$,
Prove that, $BI = GE$

Solution: Apply Menelaos to the triples (A, B, C) and (D, E, F) on the sides of triangle GHI, giving

$$\frac{HA}{AI}\frac{IB}{BG}\frac{GC}{CH} = -1, \qquad \frac{HD}{DI}\frac{IE}{EG}\frac{GF}{FH} = -1.$$

Now $AI = HD$ and $CH = GF$, so $DI = AI - AD = HD - AD = HA$ and similarly $FH = GC$; therefore

$$1 = \left(\frac{HA}{AI}\frac{IB}{BG}\frac{GC}{CH}\right)\left(\frac{HD}{DI}\frac{IE}{EG}\frac{GF}{FH}\right) = \frac{IB}{BG}\frac{IE}{EG}.$$

So $BG \cdot GE = BI \cdot IE$, or $BG(BE - BG) = BI(BE - BI)$. Since $I \neq G$, we must have $BE - BG = BI$, or $BI = GE$.

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13. Problem 2.3.3

Let ABC be a triangle, ℓ a line and L, M, N the feet of the perpendiculars to ℓ from A, B, C respectively. Prove that the perpendiculars to BC, CA, AB through L, M, N respectively, are concurrent. Their intersection is called the orthopole of the line $\ell$and the triangle ABC.

Solution: lines AL, BM, CN, which are parallel and therefore "concur". Therefore by the observation at the end of this section, the lines through BC, CA, AB perpendicular to L, M, N concur.

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14. Problem 2.4.1 (USAMO 1997/2)

Let ABC be a triangle, and draw isosceles triangles DBC, AEC, ABF external to ABC (with BC, CA, AB as their respective bases). Prove that the lines through A, B, C perpendicular to EF, FD, DE respectively, are concurrent.

Solution 1: By the observation at the end of this section it suffices to show that the lines through D, E, F perpendicular to BC, CA, AB are concurrent. But these lines are exactly the perpendicular bisectors of BC, CA, AB, which concur at the circumcenter of triangle ABC.

Solution 2: Let P be the intersection of the line through A perpendicular to EF and the line through B perpendicular to FD. Then $PE^2 - PF^2 = AE^2 - AF^2$ and $PF^2 - PD^2 = BF^2 - BD^2$, so $PE^2 - PD^2 = AE^2 - AF^2 + BF^2 - BD^2 = CE^2 - CD^2$ and PC is perpendicular to DE.

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15. Problem 2.4.2 (MOP 1997)

Let ABC be a triangle, and D, E, F the points where the incircle touches sides BC, CA, AB respectively. The parallel to AB through E meets DF at Q, and the parallel to AB through D meets EF at T. Prove that the lines CF, DE, QT are concurrent.

Solution: We want to show

$$\frac{\sin\angle TFC}{\sin\angle CFD}\frac{\sin\angle FDE}{\sin\angle EDT}\frac{\sin\angle DTQ}{\sin\angle QTF} = 1.$$

Drop perpendiculars CX, CY from C to FE, FD respectively. Then

$$\frac{\sin \angle TFC}{\sin \angle CFD} = \frac{CX/CF}{CY/CF} = \frac{CX}{CY} = \frac{CE \sin \angle XEC}{CD \sin \angle CDY} = \frac{\sin \angle AEF}{\sin \angle FDB}.$$

Since $EQ \parallel DT$, by the Law of Sines,

$$\frac{\sin \angle FDE}{\sin \angle EDT} = \frac{\sin \angle QDE}{\sin \angle QED} = \frac{QE}{QD} \quad \text{and} \quad \frac{\sin \angle DTQ}{\sin \angle QTF} = \frac{\sin \angle TQE}{\sin \angle QTE} = \frac{TE}{QE}.$$

Now $TE/QD = TF/FD = \sin \angle TDF / \sin \angle DTF = \sin \angle DFB / \sin \angle EFA$, so

$$\frac{\sin \angle TFC}{\sin \angle CFD} \frac{\sin \angle FDE}{\sin \angle EDT} \frac{\sin \angle DTQ}{\sin \angle QTF} = \frac{\sin \angle AEF}{\sin \angle FDB} \frac{QE}{QD} \frac{TE}{QE} = \frac{\sin \angle AEF}{\sin \angle FDB} \frac{\sin \angle DFB}{\sin \angle EFA} = 1$$

and DE, QT, CF concur.

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16. Problem 2.4.3 (Stanley Rabinowitz)

The incircle of triangle ABC touches sides BC, CA, AB at D, E, F, respectively. Let P be any point inside triangle ABC, and let X, Y, Z be the points where the segments PA, PB, PC respectively, meet the incircle.Prove that the lines DX, EY, FZ are concurrent.

Solution: We have

$$\frac{\sin \angle FEY}{\sin \angle YED} = \frac{FY}{YD} = \frac{YM}{YN} = \frac{\sin \angle MBY}{\sin \angle YBN} = \frac{\sin \angle ABP}{\sin \angle PBC},$$

so

$$\frac{\sin \angle FEY}{\sin \angle YED} \frac{\sin \angle EDX}{\sin \angle XDY} \frac{\sin \angle DFZ}{\sin \angle ZFE} = \frac{\sin \angle ABP}{\sin \angle PBC} \frac{\sin \angle CAP}{\sin \angle PAB} \frac{\sin \angle BCP}{\sin \angle PCA} = 1$$

and DX, EY, FZ concur.

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17. Problem 3.1.2 (MOP 1997)

Consider a triangle ABC with $AB = AC$, and points M and N on AB and AC, respectively. The lines BN and CM intersect at P. Prove that MN and BC are parallel if and only if $\angle APM = \angle APN$

Solution: First, suppose $MN \parallel BC$. Let ℓ be the bisector of angle BAC. Then as ABC and AMN are isosceles triangles, reflection in ℓ interchanges B and C, M and N. So $P = BN \cap CM$ maps to $CM \cap BN$, which is P again; therefore P must lie on ℓ and $\angle APM = \angle APN$. Conversely, suppose $\angle APM = \angle APN$. Let M' be the reflection of M in ℓ. Then the reflection of C in ℓ is $C' = AM' \cap CM$. But $AB' = AB = AC$, so we must have $B' = C$ and $M' = N$; therefore $AM = AN$ and MN is parallel to BC.

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18. Problem 3.1.4 (MOP 1996)

Let $AB_1C_1, AB_2C_2, AB_3C_3$ be directly congruent equilateral triangles. Prove that the pairwise intersections of the circumcircles of triangles $AB_1C_2, AB_2C_3, AB_3C_1$ form an equilateral triangle congruent to the first three.

Solution: Let s be the common side length of all the triangles. Let ω_i be the circumcircle of $AB_{i+1}C_{i-1}$, let O_i be the center of ω_i, and let D_i be the second intersection of ω_{i-1} and ω_{i+1}. Let $\alpha = \angle B_2AC_3$, $\beta = \angle B_3AC_1$, $\gamma = \angle B_1AC_2$. Note $\angle AD_3B_3 = \pi - \angle AC_1B_3 =$

$\pi - \angle AB_3C_1 = \angle AD_1C_1 = \angle AD_1B_3 + \angle B_3D_1C_1 = \pi - \angle AD_3B_3 + \angle C_1AB_3 = \pi + \beta - \angle AD_3B_3$, so $\angle AD_3B_3 = (\pi+\beta)/2$. Similarly $\angle AD_1C_1 = (\pi+\beta)/2$, $\angle AD_3C_3 = \angle AD_2B_2 = (\pi+\alpha)/2$, $\angle AD_2B_2 = \angle AD_1C_1 = (\pi+\gamma)/2$. Therefore $\angle B_2D_2C_2 = 2\pi - \angle B_2D_2A - \angle C_2D_2A = 2\pi - (\pi+\alpha)/2 - (\pi+\beta)/2 = (\pi+\gamma)/2$ as $\alpha+\beta+\gamma = \pi$. Consider a rotation around O_1 through $\angle AO_1B_2$. This clearly maps A to B_2, C_3 to A, and ω_1 to itself. Since distances are preserved, B_3 maps to C_2. Let ω be the circumcircle of $B_2D_2C_2$, and let P be the image of D_3. Then P lies on ω_1 as D_3 does, and P lies on ω since $\angle B_2PC_2 = \angle AD_3B_3 = (\pi+\beta)/2 = \angle B_2D_2C_2$. Since $D_3 \neq A$, $P \neq B_2$, so we must have $D_3 = D_2$. Therefore $\angle D_3O_1D_2 = \angle AO_1B_2$, so $D_2D_3 = B_2A = s$. Similarly, $D_1D_2 = D_3D_1 = s$, so triangle $D_1D_2D_3$ is congruent to the original three triangles.

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19. Problem 3.2.2 (USAMO 1992/4)

Chords $\overline{AA}, \overline{BB}, \overline{CC}$ of a sphere meet at an interior point P but are not contained in a plane. The sphere through A, B, C, P is tangent to the sphere through A', B', C', P. Prove that $\overline{AA} = \overline{BB} = \overline{CC}$

Solution: Let S be the sphere through A, B, C, and P, S' the sphere through A', B', C', and P, and O and O' the centers and r and r' the radii of S and S' respectively. Since S and S' are tangent and intersect at P, they are tangent at P, so O, O', and P are collinear with $O'P/OP = -r'/r$. Consider a homothety around P with ratio $-r'/r$. Then if X' is the image of X, $|O'X'| = |OX|r'/r$, so X lies on S if and only if X' lies on S'; therefore this homothety sends S to S'. So the image of A, which is collinear with A and P, must also lie on S', and must be A'. Similarly B' is the image of B, so $AP/PA' = BP/PB'$. Now A, B, A', B', and P are coplanar, and A, B, A', B' lie on a sphere; therefore $ABA'B'$ is a cyclic quadrilateral. So by the power-of-a-point theorem, $AP \cdot PA' = BP \cdot PB'$. Multiplying this by the equation above gives $AP = BP$, so $AA' = BB'$. Similarly $BB' = CC'$, so $AA' = BB' = CC'$.

Alternatively, we could begin by taking the cross-section through the plane containing A, B, A', B', and P. Then A, B, A', B' are concyclic, and the circle ω through A, B, and P is tangent to the circle ω' through A', B', and P, so if ℓ is their line of tangency, $\angle ABP = \angle(AP, \ell) = \angle(A'P, \ell) = \angle PB'A' = \angle BB'A' = \angle BAA' = \angle BAP$ and $AP = BP$. Similarly $A'P = B'P$, so $AA' = BB' = CC'$.

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20. Problem 3.2.4

Given three nonintersecting circles, draw the intersection of the external tangents to each pair of the circles. Show that these three points are collinear.

Solution:

Lemma: Suppose we have two noncongruent circles C_1 and C_2 whose external tangents intersect at P. Then there is a unique homothety with positive ratio sending C_1 to C_2, and its center is at P.

Proof. Any homothety with positive ratio sending C_1 to C_2 maps each of the external tangents to itself, so it maps P to itself, that is, the center must be P. Then the ratio is uniquely determined by the ratio of the radii of the two circles.

Now let C_1, C_2, C_3 be our three circles, P_i the intersection of the external tangents of C_i and C_{i+1}, and H_i the homothety with positive ratio mapping C_i to C_{i+1}. Let ℓ be the line through P_1 and P_2. Since H_i is centered at P_i by the Lemma, ℓ is fixed setwise by H_1 and H_2. Note that H_2H_1 is a homothety with positive ratio mapping C_1 to C_3; therefore it coincides with H_3^{-1}. But H_2H_1 leaves ℓ fixed, so H_3 must as well; therefore the center of H_3, P_3, must lie on ℓ. So P_1, P_2, and P_3 are collinear.

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21. Problem 4.1.1 If A, B, C, D are concyclic and $AB \cap CD = E$. Prove that,

$$\frac{AC}{BC}\frac{AD}{BD} = \frac{AE}{BE}$$

Solution: As in the proof of Theorem 4.1, triangles EAD and ECB are similar, as are triangles EAC and EDB; so $AD/BC = AE/CE$, $AC/BD = CE/BE$, and

$$\frac{AC}{BC}\frac{AD}{BD} = \frac{AE}{BE}$$

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22. Problem 4.1.2 (Mathematics Magazine, Dec. 1992)
Let ABC be an acute triangle, let H be the foot of the altitude from A, and let D, E, Q be the feet of the perpendiculars from an arbitrary point P in the triangle onto AB, AC, AH, respectively. Prove that,

$$|AB.AD - AC.AE| = BC.PQ$$

Solution: If P lies on AH, then quadrilaterals $DPHB$ and $EPHC$ are cyclic because of the right angles at D, E, and H, so $AB \cdot AD = AP \cdot AH = AC \cdot AE$, and $|AB \cdot AD - AC \cdot AE| = 0 = BC \cdot PQ$. If not, let $R = PD \cap AH$, $S = PE \cap AH$; then $DRHB$ and $ESHC$ are cyclic, so $|AB \cdot AD - AC \cdot AE| = |AR \cdot AH - AS \cdot AH| = RS \cdot AH$; since $\angle PRS = \angle DRA = \angle ABH = \angle ABC$, triangles ABC and PRS are similar, so $PQ/AH = RS/BC$ and $RS \cdot AH = BC \cdot PQ$.

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23. Problem 4.1.3
Draw tangents OA and OB from a point O to a given circle. Through A is drawn a chord AC parallel to OB; let E be the second intersection of OC with the circle.
Prove that, the line AE bisects the segment OB.
Solution: Let M be the intersection of AE with OB. Then $\angle EOM = \angle COB = \angle OCA = \angle ECA = \angle OAE = \angle OAM$, so MO is tangent to the circle through O, E, and A; therefore $MO^2 = ME \cdot MA = MB^2$ and M is the midpoint of OB.

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24. Problem 4.1.4 (MOP 1995)
Given triangle ABC, let D, E be any points on BC. A circle through A cuts the lines AB, AC, AD, AE at the points P, Q, R, S, respectively. Prove that,

$$\frac{AP.AB - AR.AD}{AS.AE - AQ.AC} = \frac{BD}{CE}$$

Solution: We will use directed distances. Let O be the center of the given circle, r its radius, and H and J the feet of the perpendiculars to BC from A and O respectively. Then by power-of-a-point, $BP \cdot BA = BO^2 - r^2$, so $AP \cdot AB = AB^2 - PB \cdot AB = AB^2 - BO^2 + r^2$. Similarly $AR \cdot AD = AD^2 - DO^2 + r^2$, so $AP \cdot AB - AR \cdot AD = (AB^2 - BO^2 + r^2) - (AD^2 - DO^2 + r^2) = AH^2 + BH^2 - BJ^2 - OJ^2 - AH^2 - DH^2 + DJ^2 + OJ^2$

$= (BH-BJ)(BH+BJ)-(DH-DJ)(DH+DJ) = HJ\cdot(BH+BJ-DH-DJ) = 2HJ\cdot BD.$
By a similar calculation $AQ \cdot AC - AS \cdot AE = 2HJ \cdot CE$, so

$$\frac{AP \cdot AB - AR \cdot AD}{AS \cdot AE - AQ \cdot AC} = \frac{2HJ \cdot BD}{2HJ \cdot EC} = \frac{BD}{EC}.$$

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25. Problem 4.1.5 (IMO 1995/1)
Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y . The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N.
Prove that the lines AM, DN, XY are concurrent.
Solution: The result is trivial if P coincides with X or Y, so suppose not. By power-of-a-point, $PB \cdot PN = PX \cdot PY = PC \cdot PM$, so quadrilateral $BCMN$ is cyclic. Then (using directed angles) $\angle MAD = \angle MAC = \pi/2 + \angle MCA = \pi/2 + \angle MCB = \pi/2 + \angle MNB = \angle MND$, so quadrilateral $ADMN$ is cyclic as well. Let $Q = AM \cap ND$, and let Y_1 and Y_2 be the intersections of QX with the circles on AC and BD respectively. Then $QX \cdot QY_1 = QA \cdot QM = QN \cdot QD = QX \cdot QY_2$, so $Y_1 = Y_2 = Y$ and Q lies on the line XY.

Alternatively, one could begin by letting $Q = AM \cap XY$. Then $QX \cdot QY = QA \cdot QM = QP \cdot QZ$ since triangles QMP and QZA are similar. This implies that Q lies on the radical axis of the circle on BD and the circumcircle of $PZDN$, namely the line ND. So AM, XY, DN concur at Q.

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26. Problem 4.2.2 (MOP 1995)
Let BB', CC' be altitudes of triangle ABC, and assume $AB \neq AC$. Let M be the midpoint of BC, H the orthocenter of ABC, and D the intersection of BC and $B'C'$.
Show that DH is perpendicular to AM.
Solution: Let AA' be the altitude from A, let N be the midpoint of AM, let ω_1 be the circle through B, C, B', and C', and let ω_2 be the circle through A, A', and M. Then A, B, A', B' are concyclic, so $HA \cdot HA' = HB \cdot HB'$; therefore H lies on the radical axis of ω_1 and ω_2. Also A', B', C', and M lie on the nine-point circle of triangle ABC, so $DB \cdot DC = DB' \cdot DC' = DA' \cdot DM$; therefore D also lies on the radical axis of ω_1 and ω_2. So DH is perpendicular to line NM, which is the same as line AM.

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27. Problem 4.2.3 (IMO 1994 proposal)
A circle ω is tangent to two parallel lines ℓ_1 and ℓ_2. A second circle ω_1 is tangent to ℓ_1 at A and to ω externally at C. A third circle ω_2 is tangent to ℓ_2 at B, to ω externally at D and to ω_1 externally at E. Let Q be the intersection of AD and BC. Prove that $QC = QD = QE$.
Solution: Let X and Y be the points where circle ω is tangent to lines ℓ_1 and ℓ_2 respectively. It is easy to check that A, C, and Y are collinear, and similarly B, D, X and A, E, B are collinear. Now $\angle CYB = \angle AYB = \angle XAY = \angle XAC = \angle AEC$, so $BECY$ is cyclic. Therefore $AC \cdot AY = AE \cdot AB$, so A lies on the radical axis of ω and ω_2. In particular, since D is their point of tangency, AD is tangent to ω and ω_2. Similarly, BC is the radical axis of ω and ω_1 and is therefore tangent to these two circles. Therefore $Q = AD \cap BC$ is the radical center of ω, ω_1, and ω_2, so QC, QD, QE are tangents and $QC = QD = QE$.

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28. Problem 4.2.4 (India, 1996)

Let ABC be a triangle. A line parallel to BC meets sides AB and AC at D and E, respectively. Let P be a point inside triangle ADE, and let F and G be the intersection of DE with BP and CP, respectively.
Show that A lies on the radical axis of the circumcircles of $\triangle PDG$ and $\triangle PFE$.

Solution: Let M be the second intersection of the circumcircle of PDG with AB and N the second intersection of the circumcircle of PFE with AC. Then $\angle MBC = \angle MDG = \angle MPG = \angle MPC$, so M, P, B, C are concyclic. Similarly, N, P, B, C are concyclic, so all of these points lie on one circle; in particular $\angle MDE = \angle MBC = \angle MNC = \angle MNE$, so quadrilateral $MNDE$ is cyclic. Since $A = AB \cap AC = MD \cap NE$, A is the radical center of $MNDE$, $MPDG$, and $NPFE$, so A lies on the radical axis of PDG and PFE.

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29. Problem 4.2.5 (IMO 1985/5)

A circle with center O passes through the vertices A and C of triangle ABC, and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangle ABC and KBN intersect at exactly two distinct points B and M. Prove that, $\angle OMB$ is a right angle.

Solution: By the radical axis theorem, AC, KN, and MB concur, at D, say. Then $\angle DMK = \angle BMK = \angle BNK = \angle CNK = \angle CAK = \angle DAK$, so D, M, A, K are concyclic. Next, let E be the second intersection of the line AM with the circle centered at O; then $\angle MEN = \angle AEN = \angle AKN = \angle AKD = \angle AMD = \angle AME$, so lines MD and EN are parallel; it therefore suffices to show $OM \perp EN$. But we also have $\angle MNE = \angle BMN = \angle BKN = \angle AKN = \angle AEN = \angle MEN$; therefore $ME = MN$, and $OE = ON$.
So, OM and EN are perpendicular.

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30. Problem 4.3.1

What do we get if we apply Brianchons theorem with three degenerate vertices?

Solution: The statement is: Let ACE be a triangle, and B, D, F the points where its inscribed circle touches sides AC, CE, EA, respectively. Then lines AD, BE, CF are concurrent.

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31. Problem 4.3.2

Let $ABCD$ be a circumscribed quadrilateral, whose incircle touches AB, BC, CD, DA at M, N, P, Q, respectively. Prove that the lines AC, BD, MP, NQ are concurrent.

Solution: Let $X = AC \cap BD$. Applying Brianchon's theorem to the degenerate hexagon $AMBCPD$, we see that lines AC, BD and MP concur, so line MP passes through point X. Similarly, applying Brianchon's theorem to $ABNCDQ$, lines AC, BD and NQ concur, so line NQ also passes through X. Hence lines AC, BD, MP, NQ concur at X.

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32. Problem 4.3.3

With the same notation (**Problem 31**), let lines BQ and BP intersect the inscribed circle at E and F, respectively. Prove that ME, NF and BD are concurrent.

Solution: Let $X = AC \cap BD$ as in the previous solution and let $Y = ME \cap NF$. By Pascal's theorem applied to hexagon $MEQNFP$, points $ME \cap NF = Y$, $EQ \cap FP = B$, $QN \cap PM = X$ are collinear; since X lies on BD, so does Y.

33. Problem 4.3.4

Let $ABCDE$ be a convex quadrilateral with $CD = DE$ and $\angle BCD = \angle DEA = \pi/2$. Let F be the point on side AB such that $AF/FB = AE/BC$. Show that, $\angle FCE = \angle FDE$ and $\angle FEC = \angle BDC$

Solution: Let $P = AE \cap BC$; then $CDEP$ is cyclic as $\angle PED = \pi/2 = \angle PCD$. Let γ be the circumcircle of $CDEP$, and let Q and R be the second intersections of DA and DB, respectively, with γ. Let $G = CQ \cap ER$; then A, G, and B are collinear by Pascal's theorem applied to hexagon $PCQDRE$. By the Law of Sines,

$$\frac{AG}{BG} = \frac{QG}{RG}\frac{\sin \angle DQC}{\sin ERD}\frac{\sin \angle RBG}{\sin \angle GAQ} = \frac{\sin \angle QRG}{\sin \angle GQR}\frac{CD}{DE}\frac{\sin \angle DBA}{\sin \angle BAD} = \frac{\sin \angle ADE}{\sin \angle CDB}\frac{AD}{BD} = \frac{AE}{BC} = \frac{AF}{BF},$$

so in fact $G = F$. Thus $\angle FCE = \angle QCE = \angle ADE$ and $\angle FEC = \angle REC = \angle BDC$.

Alternatively, define P, γ, and Q as before, and let $G = AB \cap CH$. Then $\angle AHG = \angle DHC = \angle EHD = \angle EHA$ and $\angle BCG = \angle PCH = \angle PEH = \angle AEH$

So by the Law of Sines

$$\frac{AG}{BG} = \frac{AG \sin \angle AGH}{BG \sin \angle BGC} = \frac{AH \sin \angle AHG}{BC \sin \angle BCG} = \frac{AH \sin \angle EHA}{BC \sin \angle AEH} = \frac{AE}{BC} = \frac{AF}{BF}.$$

Hence $G = F$, so $\angle FCE = \angle GCE = \angle HCE = \angle HDE = \angle ADE$.

Similarly, $\angle FEC = \angle BDC$.

This is the solutions of the Older(1999) Version of Geometry Unbound

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*This document is prepared using LaTeX

**The Diagrams are in a separate document